Homework 1

Page 83 #8A – Apply Lemma 13.2 to show that the countable collection

$$\mathbf{B} = \{(a,b) \mid a < b, \text{ a and b are rational}\}$$

is a basis that generates the standard topology on $\mathbf{R}$.

Consider the standard topology on the real line. $\mathbf{B}$ is a collection of open sets of $\mathbf{R}$ by hypothesis. Pick any open set $(c,d) \in \mathbf{R}$. $\forall\, x \in (c,d)$, we can find a rational a and b $\ni$ x $\in (a,b) \subset (c,d)$. Then applying Lemma 13.2, $\mathbf{B}$ is a basis for the standard topology on $\mathbf{R}$.

Page 91 # 6 – Show that the countable collection

$$\mathbf{D} = \{(a,b) \times (c,d) \mid a < b,\ c < d, \text{ and a,b,c, and d are rational}\}$$

is a basis for $\mathbf{R}^2$.

The countable collection $\mathbf{B} = \{(a,b) \mid a < b$, a and b are rational$\}$ and $\mathbf{C} = \{(c,d) \mid c < d$, c and d are rational$\}$ are bases of R by the 8A above. By Theorem 15.1, $\mathbf{D}$ is a basis for $\mathbf{R}^2$.

Page 100 #2 – Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Let A be closed in Y and Y be closed in X. Then A contains all its limit points with respect to Y and $A \subset Y$ by Corollary 17.7. Similarly, Y contains all its limit points with respect to X and $Y \subset X$. First, we need to show that the limit point x_0 of A with respect to X are also in Y. Then we will show that x_0 is also a Y limit point of A. (Note that it is necessary that $x_0 \in Y$ in order for it to be a Y limit point of A.) Consider such a point x_0. By definition of limit point a neighborhood U around x_0 intersects A in a point other than itself ($U \cap A - \{x_0\} \neq \{\}$). Since $A \subset Y$, it also intersects Y in a point other than itself ($U \cap Y - \{x_0\} \neq \{\}$). Then x_0 is a limit point of Y with respect to X. Hence, all X limit points of A are also X limit points of Y. Because Y is closed in X, Y contains all its limit points with respect to X (including the limit points of A with respect to X).

Now we will show that x_0 is a Y limit point of A. (Note that we cannot assume open sets in X and Y are identical). Suppose $U_Y = U_X \cap Y$ are open sets containing x_0 with $U_Y \subset Y$ and $U_X \subset X$. Since x_0 is an X limit point of A, $U_X \cap A - \{x_0\} \neq \{\}$. Then $U_X \cap A = U_X \cap A \cap Y$ (since $A \subset Y$) $= U_X \cap Y \cap A = U_Y \cap A$ (by definition of U_Y above). Then $U_X \cap A - \{x_0\} = U_Y \cap A - \{x_0\} \neq \{\}$. Hence, x_0 is a Y limit point of A. Because A is closed in Y, $x_0 \in A$ that is, A contains its X limit points. $\therefore$ A is closed in X.

Page 100 # 11 – Show that the product of two Hausdorf spaces is Hausdorf.

Let X and Y be Hausdorf spaces. Then for each pair of points $y_1 \neq y_2 \in Y$, $\exists$ disjoint neighborhoods $V_1, V_2 \subset Y \ni y_1 \in V_1$, and $y_2 \in V_2$, with $V_1 \cap V_2 = \{\}$. Consider $x_1, x_2 \in X$ (not necessarily distinct) and respective neighborhoods U_1 and U_2 around them. Consider the product space $X \times Y$. Then $(x_1,y_1) \in U_1 \times V_1$ and $(x_2,y_2) \in U_2 \times V_2$.

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \cap (V_1 \cap V_2) = (U_1 \cap U_2) \cap \{\} = \{\}$$

Then $U_1 \times V_1$ and $U_2 \times V_2$ are disjoint neighborhoods around (x_1,y_1) and (x_2,y_2), respectively. Similarly for $x_1 \neq x_2 \in X$ and $y_1, y_2 \in Y$. $\therefore$ $X \times Y$ is Hausdorf. (Note that for the first part we could have used X as the open set containing x_1 and x_2.)