Homework 2

Page 111 #1 – Prove that for the functions $f : \mathbb{R} \to \mathbb{R}$, the ε–δ definition of continuity implies the open set definition.

Let $f : X \to Y$, $X = \mathbb{R}$ and $Y = \mathbb{R}$. We must show that the ε–δ definition of continuity implies that $U = f^{-1}(V) \in X$ is open in X if V is open in Y. The ε–δ definition of continuity state that if f is continuous, $\forall\, p \in X$ and $\forall\, \varepsilon > 0$, $\exists$ a $\delta > 0 \ni$ if $x \in (p - \delta, p + \delta)$, then $f(x) \in (f(x) - \varepsilon, f(x) + \varepsilon)$. Let V be open in Y and let $p \in f^{-1}(V)$. We need to show that p is an interior point of $f^{-1}(V)$. Let $y = f(p)$. Because V is open, $\exists\, B_Y(y, \varepsilon) \subseteq V$ for some $\varepsilon > 0$. Since f is continuous at p, $\exists$ a $\delta > 0 \ni f(B_X(p, \delta)) \subseteq B_Y(y, \varepsilon)$. Hence,

$$B_X(p, \quad \subseteq f^{-1}[f(B_X(p, \delta))] \subseteq f^{-1}(B_Y(y, \varepsilon)) \subseteq f^{-1}(V)$$

and p is an interior point of $f^{-1}(V)$.

Page 111 # 2 – Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

In general yes, by Theorem 21.3 which states Let $f : X \to Y$. If the function is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n) \to f(x)$. However, there is one exception and that is when f(x) is a constant. In this case, the set Y only consists of one point and hence cannot have a limit point.

Page 111 #5 – Show that the subspace (a,b) of $\mathbb{R}$ is homeomorphic with (0,1) and the subspace [a,b] of $\mathbb{R}$ is homeomorphic with [0,1].

We need to find a homeomorphism $f : (a,b) \to (0,1)$ and $g : [a,b] \to [0,1]$. Let $a < x < b$ and $0 < y = f(x) < 1$ and the map $f : (a,b) \to (0,1)$ be

$$y = f(x) = \frac{x - a}{b - a}$$

This map is one-to-one, continuous, and has inverse $f^{-1}(y) = a + (b-a)y = x$ and hence a homeomorphism. $\therefore$ (a,b) is homeomorphic to (0,1). Use the same map for g as a homeomorphism from [a,b] to [0,1].

Page 126 # 9 – Show that the Euclidean metric d on $\mathbb{R}^n$ is a metric, as follows: If $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, define

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n),$$
$$c\mathbf{x} = (cx_1, \ldots, cx_n),$$
$$\mathbf{x} \bullet \mathbf{y} = (x_1y_1 + \ldots + x_ny_n).$$

(a) Show that $\mathbf{x} \bullet (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \bullet \mathbf{y}) + (\mathbf{x} \bullet \mathbf{z})$.
(b) Show that $|\mathbf{x} \bullet \mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$.
(c) Show that $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.
(d) Verify that d is a metric.

(a)
$$\begin{aligned}
\mathbf{x} \bullet (\mathbf{y} + \mathbf{z}) &= (x_1,\ldots,x_n) \bullet [(y_1,\ldots,y_n) + (z_1,\ldots,z_n)] \\
&= (x_1,\ldots,x_n) \bullet (y_1 + z_1,\ldots,y_n + z_n) \\
&= (x_1(y_1 + z_1) + \ldots + x_n(y_n + z_n)) \\
&= (x_1y_1 + x_1z_1 + \ldots + x_ny_n + x_nz_n) \\
&= (x_1y_1 + \ldots + x_ny_n) + (x_1z_1 + \ldots + x_nz_n) \\
&= [(x_1,\ldots,x_n) \bullet (y_1,\ldots,y_n)] + [(x_1,\ldots,x_n) \bullet [(z_1,\ldots,z_n)] \\
&= (\mathbf{x} \bullet \mathbf{y}) + (\mathbf{x} \bullet \mathbf{z})
\end{aligned}$$

(b) The statement is clearly valid if $\mathbf{x} = \mathbf{0}$. If $\mathbf{x} \neq \mathbf{0}$, let

$$\mathbf{z} = \mathbf{y} - \frac{x \bullet y}{\| x \|} x$$

Then $\mathbf{z} \bullet \mathbf{x} = 0$ and

$$\begin{aligned}
0 \le \|\mathbf{z}\|^2 = \mathbf{z} \bullet \mathbf{z} &= \left(y - \frac{x \bullet y}{\| x \|} x \right) \bullet \left(y - \frac{x \bullet y}{\| x \|} x \right) \\
&= \mathbf{y} \bullet \mathbf{y} - \mathbf{y} \bullet \frac{x \bullet y}{\| x \|^2} x - \frac{x \bullet y}{\| x \|^2} x \bullet \mathbf{y} + \frac{x \bullet y}{\| x \|^2} x \bullet \frac{x \bullet y}{\| x \|^2} x \\
&= \|\mathbf{y}\|^2 - 2\frac{(x \bullet y)^2}{\| x \|^2} + \frac{(x \bullet y)^2}{\| x \|^4} x \bullet x \\
&= \|\mathbf{y}\|^2 - 2\frac{(x \bullet y)^2}{\| x \|^2} + \frac{(x \bullet y)^2}{\| x \|^4} \| x \|^2 \\
&= \|\mathbf{y}\|^2 - 2\frac{(x \bullet y)^2}{\| x \|^2} + \frac{(x \bullet y)^2}{\| x \|^2}
\end{aligned}$$

$$0 \le \|\mathbf{y}\|^2 - \frac{(x \bullet y)^2}{\| x \|^2}$$

Hence,

$$(\mathbf{x} \bullet \mathbf{y})^2 \le \|\mathbf{y}\|^2 \|\mathbf{x}\|^2 \text{ and}$$
$$|\mathbf{x} \bullet \mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|.$$

(c) $\|\mathbf{x} + \mathbf{y}\|^2 = [(\mathbf{x} + \mathbf{y}) \bullet (\mathbf{x} + \mathbf{y})] = [\mathbf{x}\bullet\mathbf{x} + \mathbf{x}\bullet\mathbf{y} + \mathbf{y}\bullet\mathbf{x} + \mathbf{y}\bullet\mathbf{y}]$

$= [|\mathbf{x}\bullet\mathbf{x} + 2\,\mathbf{x}\bullet\mathbf{y} + \mathbf{y}\bullet\mathbf{y}|] \leq \|\mathbf{x}\| + 2\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\| = [\|\mathbf{x}\| + \|\mathbf{y}\|]^2$ using part b.

Hence, $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ since both sides are positive

(d) To show that the Euclidean norm is a metric, we have to show that if $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$

(i) $d(\mathbf{x},\mathbf{y}) \geq 0$ with $d(\mathbf{x},\mathbf{y}) = 0$ only when $\mathbf{x} = \mathbf{y}$

(ii) $d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x})$

(iii) $d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}) \geq d(\mathbf{x},\mathbf{z})$

(i) by definition $d(\mathbf{x},\mathbf{y}) \geq 0$ (it is the positive square root and if $\mathbf{x} \neq \mathbf{y}$ then at least on pair x_i and y_i are distinct and their difference squared is positive)
$d(\mathbf{x},\mathbf{x}) = \|\mathbf{x} - \mathbf{x}\| = \|\mathbf{0}\| = \|\mathbf{0}\| = (0^2 + \ldots + 0^2)^{1/2} = 0$

(ii) $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \|(x_1 - y_1,\ldots,x_n - y_n)\| = [(x_1 - y_1)^2,\ldots,(x_n - y_n)^2]^{1/2}$
$= [(y_1 - x_1)^2,\ldots,(y_n - x_n)^2]^{1/2} = \|(y_1 - x_1,\ldots,y_n - x_n)\| = \|\mathbf{y} - \mathbf{x}\| = d(\mathbf{y},\mathbf{x})$

(iii) $d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}) = \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\| \geq \|(\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z})\|$ by part c
$= \|\mathbf{x} - \mathbf{y} + \mathbf{y} - \mathbf{z}\| = \|\mathbf{x} - \mathbf{z}\|$
Hence, $\|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\| \geq \|\mathbf{x} - \mathbf{z}\|$ and $d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}) \geq d(\mathbf{x},\mathbf{z})$

Page 135 # 11d – Given a sequence of functions $f_n : X \to \mathsf{R}$, let

$$s_n = \sum_{i=1}^{n} f_i(x)$$

Prove the Weierstrass M-test for uniform convergence: If $|f_i(x)| \leq M_i$ for all $x \in X$ and all *i*, and if the series $\sum M_i$ converges, then the sequence (s_n) converges uniformly to a function s.

Assume that $|f_i(x)| \leq M_i$ for all $x \in X$ and all *i* and that the series $\sum M_i$ converges,
Let $r_n = \sum_{i=n+1}^{\infty} M_i$ using the hypothesis. Part (c) guarantees that the series
$s_n = \sum_{i=1}^{n} f_i(x)$ converges, that is, $n \to \infty$, then $s_n(x) \to s(x)$.

Consider the case when k > n. Then

$$s_k(x) - s_n(x) = \sum_{i=1}^{k} f_i(x) - \sum_{i=1}^{n} f_i(x)$$
$$s_k(x) - s_n(x) = f_{n+1}(x) + \ldots + f_k(x)$$
$$|s_k(x) - s_n(x)| \leq |f_{n+1}(x)| + \ldots + |f_k(x)|$$
$$|s_k(x) - s_n(x)| \leq M_{n+1} + \ldots + M_k$$

$$|s_k(x) - s_n(x)| \leq M_{n+1} + \ldots$$

$$|s_k(x) - s_n(x)| \leq \sum_{i=n+1}^{\infty} M_i = r_n$$

$$\lim_{k\to\infty} |s_k(x) - s_n(x)| \leq \lim_{k\to\infty} \sum_{i=n+1}^{\infty} M_i = \lim_{k\to\infty} r_n$$

$$|s(x) - s_n(x)| \leq \sum_{i=n+1}^{\infty} M_i = r_n$$

Hence,

$$|s(x) - s_n(x)| \leq r_n$$