Homework 3

Page 152 # 2 – Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \{\}$ for all n. Show that $\bigcup A_n$ is connected.

This proof will be done by contradiction. Let $\{A_n\}$ be a sequence of connected subspaces of X. Assume that $A=\bigcup A_n$ is not connected, ie. A has a separation consisting two sets C and D. Consider and arbitrary $A_i \in \{A_n\}$. By Lemma 23.2, A_i lies entirely within either C or D. The there are two cases to consider: 1) All the A_i are entirely in either C or they are all entirely in D. or 2) Some are entirely in C and some are entirely in D. In case 1, if they are all in C, then C and D are not a separation of A which is a contradiction. In case 2, if they are split then the A_i's in C are disjoint from the A_i's in D. This contradicts the hypothesis $A_n \cap A_{n+1} \neq \{\}$ for all n.

$\therefore$ The $\bigcup A_n$ is connected.

Page 158 # 3 – Let f: $X \rightarrow X$ be continuous. Show that if X = [0,1], there is a point x such that f(x) = x. The point x is called a fixed point of f. What happens if X equals [0,1) or (0,1)?

X is a connected space and an ordered set in the order topology. Consider two points a,b $\in$ X. Choose the midpoint x_1 that is between f(a) and f(b). Then by the intermediate value theorem, $\exists$ a point $c_1 \in [a,b]$ such that $f(c_1) = x_1$. If $c_1 = x_1$, we have found a fixed point. If not, choose the midpoint x_2 between c_1 and x_1. Then $\exists$ a point $c_2 \in [c_1, x_1]$ or $[x_1, c_1]$such that $f(c_2) = x_2$. If $c_2 = x_2$. Continue in this fashion. There will then be two convergent sequences of points $c_1, c_2, \ldots$ and $x_1, x_2, \ldots$ with the property $|c_n - x_n| \rightarrow 0$ as $n \rightarrow \infty$ because f continuous on X. In other words $|x_n - f(x_n)| = |f(c_n) - f(x_n)| \rightarrow 0$. Hence, in the limit f(x) = x.

Not true for X equals [0,1) or (0,1) because f is not uniformly continuous on these intervals for all f. For example, f(x) = (x+1)/2 has a fixed point x = 1. However, this has no fixed point on the interval [0,1) or (0,1)

Page 170 # 3 – Show that a finite union of compact subspaces of X is compact.

Consider a collection $\{C_1,\ldots,C_n\}$ of compact subspaces of X. Let $C = \bigcup_{i=1}^{n} C_i$. Let A be an arbitrary cover for C. Then each of the C_k has a finite subcover A_k by the definition of compact. Then C will also have a finite subcover $B = \bigcup_{i=1}^{n} A_i$.

$\therefore$ C is compact.

Lemma 26.4 – If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exists disjoint open sets U and V of X containing x_0 and Y, respectively.

Let Y be a compact subspace of the Hausdorf space X and let $x_0 \in X - Y$. Let $y \in Y$. The points x_0 and y are distinct. By the Hausdorf conditions, for each point y, $\exists$ disjoint open sets U_y and V_y around containing x_0 and y, respectively. The collection $\{V_y \mid y \in Y\}$ is a covering of Y by sets open in X. A finite collection of them $\{V_1, \ldots, V_n\}$ covers Y because Y is compact. The open set $V = V_1 \cup \ldots \cup V_n$ contains Y and is disjoint from the open set $U = U_1 \cap \ldots \cap U_n$ which contains x_0 because it is the intersection of the corresponding open sets around x_0.

$\therefore$ there exists disjoint open sets U and V of X containing x_0 and Y, respectively.

Page 170 # 5 – Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exists disjoint open sets U and V containing A and B, respectively.

Let A and B be disjoint compact subspaces of the Hausdorff space X. Let x be an arbitrary points of A. The points of x is not in Y because A and B are disjoint. Then by Lemma 26.4 $\exists$ disjoint open sets U_x and V_x such that $x \in U_x$ and V_x contains B. The collection $\{U_x\}$ is a covering of A by sets open in X. A finite collection of them $\{U_1, \ldots, U_n\}$ covers A because A is compact. The open set $U = U_1 \cup \ldots \cup U_n$ contains A and is disjoint from $V = V_1 \cap \ldots \cap V_n$ which contains B because it is the finite intersection of the corresponding open sets containing B.

$\therefore$ there exists disjoint open sets U and V containing A and B, respectively.